

On canonical bases and induction of W -graphs

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A canonical basis in the sense of Lusztig is a basis of a free module over a ring of Laurent polynomials that is invariant under a certain semilinear involution and is obtained from a fixed "standard basis" through base change matrix with polynomial entries whose constant terms equal the identity matrix.

Among the better known examples of canonical bases are the Kazhdan-Lusztig basis of Iwahori-Hecke algebras (see [12]), Lusztig's canonical basis of quantum groups and the Howlett-Yin basis of induced W -graph modules (see [10] and [11]).

This paper has two major theoretical goals: First to show that having bases is superfluous in the sense that canonicalisation can be generalized to non-free modules. This construction is functorial in the appropriate sense. The second goal is to show that Howlett-Yin-induction of W -graphs is a functor between module categories of W -graph-algebras that satisfies all the properties one hopes for when a functor is called "induction", namely transitivity and a Mackey formula.

These insights will then be applied to give an affirmative answer to a conjecture from the author's thesis.

1 Introduction

The ring $\mathbb{Z}[v^{\pm 1}]$ of Laurent polynomials has an involutive automorphism $\overline{}$ defined by $\overline{v} := v^{-1}$.

If M is a free $\mathbb{Z}[v^{\pm 1}]$ -module equipped with an $\overline{}$ -semilinear involution ι and a "standard" basis $(t_x)_{x \in X}$ then a *canonical basis w.r.t. (t_x) and ι* in the sense of Lusztig is a basis (c_x) of M such that $\iota(c_x) = c_x$ and $c_x \in t_x + \sum_{y \in X} v\mathbb{Z}[v]t_y$ hold.

Kazhdan and Lusztig showed in [12] that the Iwahori-Hecke algebra of any Coxeter group (W, S) has a canonical basis w.r.t. the standard basis $(T_w)_{w \in W}$ and the involution $\iota(T_w) = T_{w^{-1}}^{-1}$ which is now known simply as the Kazhdan-Lusztig basis. (Note that Lusztig used a slightly different construction in [13] which essentially replaces

$v\mathbb{Z}[v]$ with $v^{-1}\mathbb{Z}[v^{-1}]$ though this does not change results significantly) The special features of the action of Hecke algebra on itself w.r.t. this basis are captured in the definition of W -graphs in the same paper.

In [10] Howlett and Yin showed that given any parabolic subgroup $W_J \leq W$ and a W_J -graph (\mathfrak{C}, I, m) representing the H_J -module V , then the induced module $\text{Ind}_{H_J}^H(V) := H \otimes_{H_J} V$ is also represented by a W -graph which they constructed explicitly in terms of a canonical basis of $\text{Ind}_{H_J}^H(V)$. They developed their ideas of inducing W -graphs further in [11].

In [7] Gyoja proved that given any *finite* Coxeter group (W, S) all complex representations of the Hecke-algebra can in fact be realized by a W -graph. His proof was not constructive but introduced the W -graph algebra which I investigated further in my thesis [8] and in my previous paper [9]. The fundamental property of the W -graph algebra Ω is that the Hecke algebra is canonically embedded into $\mathbb{Z}[v^{\pm 1}]\Omega$, any representation of H given by a W -graph canonically extends to a representation of Ω and vice versa. In this sense W -graphs can (and I'm advocating that they should) be understood not as combinatorial objects encoding certain matrices but as modules of an algebra.

This paper is organised as follows: Section 2 is about modules over (generalized) Laurent polynomial rings equipped with an $\overline{}$ -semilinear involution. It defines canonical modules and canonicalisations of modules. The main theorem in this section is theorem 2.7 which proves a sufficient condition to recognize canonical modules and also shows that under the conditions present in the context of Hecke algebras (though no reference to Hecke algebras is made in this section) the canonicalisation is unique and functorial w.r.t. positive maps.

Section 3 recalls the definition of Iwahori-Hecke algebras, W -graphs and W -graph algebras.

Section 4 proves the main theorem of this paper that Howlett-Yin-induction is well-defined as a map $\Omega_J\text{-}\mathbf{mod} \rightarrow \Omega\text{-}\mathbf{mod}$. The proof is inspired by Lusztig's elegant treatment of the μ -values in [13] instead of the more laborious proof in Howlett and Yin's paper although its greater generality comes at the price of being only slightly shorter than Howlett and Yin's proof. On the other hand the proof in the style of Lusztig has the additional bonus that it provides an algorithm to compute p -polynomials and μ -values without having to compute r -polynomials. This algorithm is made explicit in 4.7. As an application it is proven that the W -graph algebra associated to a parabolic subgroup $W_J \leq W$ can be canonically identified with a subalgebra of the W -graph algebra of W .

Section 5 then proves that Howlett-Yin-induction has all the nice properties one expects: It is indeed a functor between those module categories, a concrete presentation of this functor is given, transitivity of induction and a Mackey-theorem are proven.

Section 6 then applies these findings. An improved algorithm to compute μ -values is given which makes the simultaneous computation of p -polynomials less necessary (though not completely avoidable) thus improving on the performance of existing algorithms. Additionally a very short proof of a result of Geck on induction of Kazhdan-Lusztig cells (from [4]) is given.

2 Canonicalisation of modules

Fix a commutative ring k and a totally ordered, additively written group Γ . Consider the ring $\mathcal{A} := k[\Gamma]$ where we write the group elements in \mathcal{A} as v^γ so that \mathcal{A} becomes the ring of "generalized Laurent polynomials in v " with coefficients in k . This k -algebra has an involutive automorphism $\overline{}$ defined by $\overline{v^\gamma} := v^{-\gamma}$.

We consider the smash product $\widehat{\mathcal{A}} := \mathcal{A} \rtimes \langle \iota \rangle$ where $\langle \iota \rangle$ is a cyclic group of order two acting as $\overline{}$ on \mathcal{A} .¹ An $\widehat{\mathcal{A}}$ -module is thus the same as an \mathcal{A} -module M equipped with an $\overline{}$ -semilinear involution $\iota : M \rightarrow M$.

2.1 Definition:

Let M be an arbitrary k -module. The scalar extension $\mathcal{A} \otimes_k M$ is naturally an \mathcal{A} -module and via $\iota(a \otimes m) := \overline{a} \otimes m$ it is also an $\widehat{\mathcal{A}}$ -module which will be denoted by \widehat{M} . Any $\widehat{\mathcal{A}}$ -module V that is isomorphic to \widehat{M} for some $M \in k\text{-mod}$ is called a *canonical module* and any $\widehat{\mathcal{A}}$ -module isomorphism $c : \widehat{M} \rightarrow V$ is called a *canonicalisation of V* . If M is free and $(b_x)_{x \in X}$ is a basis of M , then the image of this basis under a canonicalisation c is called the *canonical basis of V* associated to $(b_x)_{x \in X}$ and c .

2.2: Note that $M \mapsto \widehat{M}$ and $f \mapsto \text{id}_{\mathcal{A}} \otimes f$ is a (faithful) functor $k\text{-mod} \rightarrow \widehat{\mathcal{A}}\text{-mod}$.

2.3 Example:

Obviously most $\widehat{\mathcal{A}}$ -modules are not canonical. For example the only canonical $\widehat{\mathcal{A}}$ -module that is finitely generated over k is the zero module. Hence $V = k[i] = k[x]/(x^2 + 1)$ is not canonical where v operates as multiplication by i and ι operates as $i \mapsto -i$. Therefore the question arises how one can recognize if a given module is canonical and how one can find a canonicalisation.

An obvious restatement of the definition is the following:

2.4 Proposition:

Let V be an arbitrary $\widehat{\mathcal{A}}$ -module. Then V is canonical iff there exists k -submodule M of V such that

- a.) $V = \bigoplus_{\gamma \in \Gamma} v^\gamma M$ as a k -module.
- b.) ι operates as 1 on M , i.e. $\iota \cdot m = m$ for all $m \in M$.

In this case $c : \widehat{M} \rightarrow V, a \otimes m \mapsto am$ is a canonicalisation.

¹ Remember that given any k -algebra A , monoid G , and any monoid homomorphism $\phi : G \rightarrow (\text{End}(A), \circ)$ the algebra $A \rtimes_\phi G$ is defined as the k -algebra that has $A \otimes_k k[G]$ as its underlying k -module and extends the multiplication of A and $k[G]$ via $(a \otimes g) \cdot (b \otimes h) := a\phi(g)(b) \otimes gh$. It is also denoted $A \rtimes G$ if the morphism ϕ is understood.

2.5 Definition:

Let V be a $\widehat{\mathcal{A}}$ -module and (X, \leq) a poset. A X -graded shadow of M is a collection $(M_x)_{x \in X}$ of k -submodules of V such that

- a.) $V = \bigoplus_{\substack{x \in X \\ \gamma \in \Gamma}} v^\gamma M_x$ as a k -module and
- b.) $\iota \cdot m_z \in m_z + \sum_{y < z} \mathcal{A}M_y$ for all $m_z \in M_z$

In light of the above proposition a shadow is something like a canonicalisation "up to lower order error terms". The next theorem shows that these error terms can be corrected by a "triangular base change" if the poset satisfies a finiteness condition. It therefore provides a sufficient criterion for the existence of a canonicalisation which is inspired by the construction of the the Kazhdan-Lusztig basis of Iwahori-Hecke algebras and the Howlett-Yin basis of induced W -graph modules. First we need a lemma.

2.6 Lemma:

Let V be a $\widehat{\mathcal{A}}$ -module, (X, \leq) a partially ordered set and $(M_x)_{x \in X}$ a X -graded shadow of V . If $X_0 \subseteq X$ is finite, $f_x \in \mathcal{A}_{>0}M_x$ for $x \in X_0$ and $f = \sum_{x \in X_0} f_x$ satisfies $\iota \cdot f = f$, then $f_x = 0$ for all $x \in X_0$.

Proof. Assume the contrary. Wlog we can also assume $f_x \neq 0$. Otherwise we could just shrink the set X_0 . Let $X_1 \subseteq X_0$ be the subset of all maximal elements of X_0 . This is a non-empty subset because X_0 is non-empty and finite. Thus

$$f \in \sum_{x \in X_1} \mathcal{A}_{>0}M_x + \sum_{\substack{y \in X \\ \exists x \in X_1: y < x}} \mathcal{A}M_y$$

Now let $f_x = \sum_{i \in I_x} a_{ix} m_{ix}$ for $a_{ix} \in \mathcal{A}_{>0}$ and $m_{ix} \in M_x$. Then

$$\begin{aligned} \iota f &\in \sum_{x \in X_1, i \in I_x} \iota(a_{ix} f_{ix}) + \sum_{\substack{y \in X \\ \exists x \in X_1: y < x}} \mathcal{A} \iota M_y \\ &= \sum_{x \in X_1, i \in I_x} \overline{a_{ix}} \iota(f_{ix}) + \sum_{\substack{y \in X \\ \exists x \in X_1: y < x}} \mathcal{A} \iota M_y \\ &= \sum_{x \in X_1, i \in I_x} \overline{a_{ix}} f_{ix} + \sum_{\substack{y \in X \\ \exists x \in X_1: y < x}} \mathcal{A} M_y \\ &\subseteq \sum_{x \in X_1} \mathcal{A}_{<0}M_x + \sum_{\substack{y \in X \\ \exists x \in X_1: y < x}} \mathcal{A} M_y \end{aligned}$$

Comparing the x -components of f and ιf for $x \in X_1$ we find $f_x \in \mathcal{A}_{>0}M_x \cap \mathcal{A}_{<0}M_x = 0$ contrary to the assumption $f_x \neq 0$. \square

2.7 Theorem:

Let (X, \leq) be a poset such that $(-\infty, y] := \{x \in X \mid x \leq y\}$ is finite for all $y \in X$.

- a.) If an $\widehat{\mathcal{A}}$ -module V has a X -graded shadow $(M_x)_{x \in X}$ then it is canonical and there exists a unique canonicalisation $c : \widehat{M} \rightarrow V$ where $M := \bigoplus_{x \in X} M_x$ such that $c(m) \in m + \mathcal{A}_{>0}M$ for all $m \in M$.

More precisely it satisfies $c(m_x) \in m_x + \sum_{y < x} \mathcal{A}_{>0}M_y$ for all $m_x \in M_x$.

- b.) The canonicalisation above depends functorially on the shadow w.r.t. *positive maps*. More precisely let V_1, V_2 be two $\widehat{\mathcal{A}}$ -modules with X -graded shadows $(M_{i,x})_{x \in X}$ and canonicalizations $c_i : \widehat{M}_i \rightarrow V_i$ and let $\phi : V_1 \rightarrow V_2$ be a $\widehat{\mathcal{A}}$ -linear map with $\phi(\mathcal{A}_{\geq 0}M_1) \subseteq \mathcal{A}_{\geq 0}M_2$.

There is an induced map $M_1 = \mathcal{A}_{\geq 0}M_1 / \mathcal{A}_{>0}M_1 \xrightarrow{\phi} \mathcal{A}_{\geq 0}M_2 / \mathcal{A}_{>0}M_2 = M_2$ and this induces an $\widehat{\mathcal{A}}$ -linear map $\widehat{\phi} : \widehat{M}_1 \rightarrow \widehat{M}_2$. This map satisfies $c_2 \circ \widehat{\phi} = \phi \circ c_1$ holds, i.e. the diagram 1 commutes.

$$\begin{array}{ccc}
 \widehat{M}_1 & \xrightarrow{\widehat{\phi}} & \widehat{M}_2 \\
 c_1 \downarrow \cong & & c_2 \downarrow \cong \\
 V_1 & \xrightarrow{\phi} & V_2
 \end{array}$$

Figure 1: Functoriality of canonicalisation of shadows

Before we begin the proof observe that any k -linear map $f : M_z \rightarrow \mathcal{A}M_y$ can be written as $f(m_z) = \sum_{\gamma \in \Gamma} v^\gamma f_\gamma(m_z)$ with uniquely determined k -linear maps $f_\gamma : M_z \rightarrow M_y$ that have the property that $\{\gamma \mid f_\gamma(m_z) \neq 0\}$ is finite for each $m_z \in M_z$ so that the sum is indeed well-defined. Having this way of writing such maps in mind we can define $\overline{f} : M_z \rightarrow \mathcal{A}M_y$ to be the map $\overline{f}(m_z) := \sum_{\gamma \in \mathbb{Z}} v^{-\gamma} f_\gamma(m_z)$. We will use this notation for the proof to simplify the notation.

Proof. The uniqueness of c follows from the above lemma because if $c, c' : \widehat{M} \rightarrow V$ are two canonicalisations satisfying the stated property then $f := c(1 \otimes m) - c'(1 \otimes m)$ is an element of $\mathcal{A}_{>0}M$ with $\iota \cdot f = f$ so that $f = 0$ by the lemma.

Concerning the existence consider the k -linear maps $\rho_{yz} : M_z \rightarrow \mathcal{A}M_y$ defined by

$$\forall m_z \in M_z : \iota \cdot m_z = \sum_y \rho_{yz}(m_z).$$

By assumption $\rho_{yz} = 0$ unless $y \leq z$ and $\rho_{zz}(m_z) = m_z$.

Following the usual construction of the Kazhdan-Lusztig polynomials and R -polynomials we will recursively construct k -linear maps $\pi_{yz} : M_z \rightarrow \mathcal{A}_{\geq 0}M_y$ such that:

- $\pi_{yz} = 0$ unless $y \leq z$ and $\pi_{zz}(m_z) = m_z$
- $\pi_{xz} = \sum_{x \leq y \leq z} \rho_{xy} \circ \overline{\pi_{yz}}$

The first step is to observe

$$\sum_{x \leq y \leq z} \rho_{xy} \circ \overline{\pi_{yz}} = \begin{cases} \text{id}_{M_x} & \text{if } x = z \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

This follows from the fact that ι has order two:

$$\begin{aligned} m_z &= \iota \cdot \iota \cdot m_z \\ &= \sum_{y \leq z} \iota \cdot \underbrace{\rho_{yz}(m_z)}_{\in \mathcal{AM}_y} \\ &= \sum_{x \leq y \leq z} (\rho_{xy} \circ \overline{\pi_{yz}})(m_z) \end{aligned}$$

Fix $z \in X$. Define $\pi_{zz}(m_z) := m_z$ and $\pi_{xz} := 0$ for all $x \not\leq z$. If $x < z$ then assume inductively that π_{yz} is already known for all $x < y \leq z$. It is therefore possible to define

$$\alpha_{xz} := \sum_{x < y \leq z} \rho_{xy} \circ \overline{\pi_{yz}}.$$

This map satisfies

$$\begin{aligned} \alpha_{xz} &= \sum_{x < y \leq z} \rho_{xy} \circ \overline{\sum_{y \leq w \leq z} \rho_{yw} \circ \overline{\pi_{wz}}} \\ &= \sum_{x < y \leq w \leq z} \rho_{xy} \circ \overline{\rho_{yw}} \circ \pi_{wz} \\ &= \sum_{x < w \leq z} \left(\sum_{x < y \leq w} \rho_{xy} \circ \overline{\rho_{yw}} \right) \circ \pi_{wz} \\ &= \sum_{x < w \leq z} (0 - \rho_{xx} \circ \overline{\rho_{xw}}) \circ \pi_{wz} \\ &= \sum_{x < w \leq z} -\overline{\rho_{xw}} \circ \pi_{wz} \\ &= -\overline{\alpha_{xz}} \end{aligned}$$

Therefore we obtain $\alpha_0 = 0$ in the decomposition $\alpha_{xz} = \sum_{\gamma \in \Gamma} v^\gamma \alpha_\gamma$. Now define $\pi_{xz} := \sum_{\gamma > 0} v^\gamma \alpha_\gamma$ so that $\alpha_{xz} = \pi_{xz} - \overline{\pi_{xz}}$ holds. This shows that $\pi_{xz} \in \mathcal{A}_{>0}$ as well as

$$\sum_{x \leq y \leq z} \rho_{xy} \circ \overline{\pi_{yz}} = \rho_{xx} \circ \overline{\pi_{xz}} + \alpha_{xz} = \overline{\pi_{xz}} + \alpha_{xz} = \pi_{xz}.$$

Thus the existence of all π_{xz} is established and we can define the \mathcal{A} -linear map $c : \widehat{M} \rightarrow M$ by

$$\forall m_z \in M_z : c(1 \otimes m_z) := \sum_{x \leq z} \pi_{xz}(m_z) = m_z + \sum_{x < z} \pi_{xz}(m_z).$$

It is bijective because it is "upper triangular with unit diagonal". The map is also $\widehat{\mathcal{A}}$ -linear because of

$$\begin{aligned} \iota \cdot \pi(1 \otimes m_z) &= \sum_{y \leq z} \iota \cdot \pi_{yz}(m_z) \\ &= \sum_{x \leq y \leq z} \rho_{xy} \overline{\pi_{yz}}(m_z) \\ &= \sum_{x \leq z} \pi_{xz}(m_z) \\ &= \pi(1 \otimes m_z) \\ &= \pi(\iota \cdot (1 \otimes m_z)) \end{aligned}$$

Finally we have to show that c is functorial. Let M_1, M_2, ϕ be as in the statement of the theorem and fix an arbitrary $m_1 \in M_1$. Then $c_1(m_1) \in m_1 + \mathcal{A}_{>0}M_1$ so that $\phi(c_1(m_1)) \in \phi(m_1) + \mathcal{A}_{>0}M_2$. Also $\widehat{\phi}(m_1) \in \phi(m_1) + \mathcal{A}_{>0}M_2$ by construction of $\widehat{\phi}$ so that $c_2(\widehat{\phi}(m_1)) \in \widehat{\phi}(m_1) + \mathcal{A}_{>0}M_2 = \phi(m_1) + \mathcal{A}_{>0}M_2$. Therefore $f := \phi(c_1(m_1)) - c_2(\widehat{\phi}(m_1)) \in \mathcal{A}_{>0}M_2$. Additionally, since all four maps are $\widehat{\mathcal{A}}$ -linear and $m_1 \in \widehat{M}_1$ is ι -invariant f satisfies $\iota f = f$ so that lemma 2.6 implies $f = 0$. This proves the commutativity of the diagram. \square

2.8 Corollary:

Let (X, \leq) be a poset such that $\{x \in X \mid x \leq y\}$ is finite for all $y \in X$. Furthermore let V be an $\widehat{\mathcal{A}}$ -module, U an $\widehat{\mathcal{A}}$ -submodule of V and $(M_x)_{x \in X}$ an X -graded shadow for V . Define $N_x := U \cap M_x$ for all $x \in X$.

If U is generated as an \mathcal{A} -module by $\sum_{x \in X} N_x$, then (N_x) is an X -graded shadow for U , (M_x/N_x) is an X -graded shadow for V/U , the canonicalisation $\widehat{M} \rightarrow V$ restricts to the canonicalisation $\widehat{N} \rightarrow U$ and induces the canonicalisation $\widehat{M/\widehat{N}} \rightarrow V/U$ on the quotients.

Proof. This follows immediately from functoriality of canonicalisation applied to the embedding $U \hookrightarrow V$ and the quotient map $V \rightarrow V/U$ respectively. \square

2.9: In terms of canonical bases this corollary recovers the theorem that if $(t_x)_{x \in X}$ is an \mathcal{A} -basis for V and U is spanned as an \mathcal{A} -module by a subset $(t_x)_{x \in Y}$ of that basis, then the canonical basis for U is the subset $(c_x)_{x \in Y}$ of the canonical basis $c_x := c(t_x)$ of M and the canonical basis of the quotient V/U is spanned by the vectors $(c_x)_{x \in X \setminus Y}$ (more precisely by their images under the quotient map $V \rightarrow V/U$).

2.10: An important special case of this corollary is the case where U is of the form $U = \sum_{x \in I} \mathcal{A}M_x$ for some order ideal $I \trianglelefteq X$ (i.e. a subset with the property $x \in I \wedge y \leq x \implies y \in I$). Note that all such U are $\widehat{\mathcal{A}}$ -submodules by definition of X -graded shadows.

3 Hecke algebras, W-graphs and W-graph algebras

For the rest of the paper fix a (not necessarily finite) Coxeter group (W, S) , a totally ordered, additive group Γ and a weight function $L : W \rightarrow \Gamma$, i.e. a function with $l(xy) = l(x) + l(y) \implies L(xy) = L(x) + L(y)$. We will use the shorthand $v_s := v^{L(s)} \in \mathbb{Z}[\Gamma]$ and the standard assumption $L(s) > 0$ for all $s \in S$.

3.1 Definition (c.f. [5]):

The *Iwahori-Hecke algebra* $H = H(W, S, L)$ is the $\mathbb{Z}[\Gamma]$ -algebra which is freely generated by $(T_s)_{s \in S}$ subject only to the relations

$$T_s^2 = 1 + (v_s - v_s^{-1})T_s \quad \text{and}$$

$$\underbrace{T_s T_t T_s \dots}_{m_{st} \text{ factors}} = \underbrace{T_t T_s T_t \dots}_{m_{st} \text{ factors}}$$

where m_{st} denotes the order of $st \in W$.

For each parabolic subgroup $W_J \leq W$ the Hecke algebra $H(W_J, J, L|_{W_J})$ will be identified with the *parabolic subalgebra* $H_J := \text{span}_{\mathbb{Z}[\Gamma]} \{ T_w \mid w \in W_J \} \subseteq H$.

3.2 Definition (c.f. [12] and [5]):

A *W-graph with edge weights in k* is a triple (\mathfrak{C}, I, m) consisting of a finite set \mathfrak{C} of vertices, a vertex labelling map $I : \mathfrak{C} \rightarrow \{J \mid J \subseteq S\}$ and a family of edge weight matrices $m^s \in k^{\mathfrak{C} \times \mathfrak{C}}$ for $s \in S$ such that the following conditions hold:

- a.) $\forall x, y \in \mathfrak{C} : m_{xy}^s \neq 0 \implies s \in I(x) \setminus I(y)$.
- b.) The matrices

$$\omega(T_s)_{xy} := \begin{cases} -v_s^{-1} \cdot 1_k & \text{if } x = y, s \in I(x) \\ v_s \cdot 1_k & \text{if } x = y, s \notin I(x) \\ m_{xy}^s & \text{otherwise} \end{cases}$$

induce a matrix representation $\omega : k[v^{\pm 1}]H \rightarrow k[v^{\pm 1}]^{\mathfrak{C} \times \mathfrak{C}}$.

The associated directed graph is defined as follows: The vertex set is \mathfrak{C} and there is a directed edge $x \leftarrow y$ iff $m_{xy}^s \neq 0$ for some $s \in S$. If this is the case, then the value m_{xy}^s is called the *weight* of the edge. The set $I(x)$ is called the *vertex label* of x .

Note that condition 1 and the definition of $\omega(T_s)$ already guarantees $\omega(T_s)^2 = 1 + (v - v^{-1})\omega(T_s)$ so that the only non-trivial requirement in condition 2 is the braid relation $\omega(T_s)\omega(T_t)\omega(T_s) \dots = \omega(T_t)\omega(T_s)\omega(T_t) \dots$.

Given a W -graph as above the matrix representation ω turns $k[\Gamma]^{\mathfrak{C}}$ into a module for the Hecke algebra. It is natural to ask whether a converse is true. In many situations the answer is yes as shown by Gyoja.

3.3 Theorem (c.f. [7]):

Let W be finite, $K \subseteq \mathbb{C}$ be a splitting field for W and assume $\Gamma = \mathbb{Z}$ and $L(s) = 1$ for all $s \in S$. Then every irreducible representation of $K(v)H$ can be realized as a W -graph module for some W -graph with edge weights in K .

Gyoja also provides an example of a finite-dimensional representation of the affine Weyl group of type \tilde{A}_n that is not induced by a W -graph.

3.4 Definition:

Assume $\Gamma = \mathbb{Z}$ and consider the free algebra $\mathbb{Z}\langle e_s, x_{s,\gamma} | s \in S, -L(s) < \gamma < L(s) \rangle$. Define

$$\iota(T_s) := -v_s^{-1}e_s + v_s(1 - e_s) + \sum_{-L(s) < \gamma < L(s)} v^\gamma x_{s,\gamma} \in \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} \mathbb{Z}\langle e_s, x_{s,\gamma} \rangle$$

for all $s \in S$ and write

$$\sum_{\gamma \in \Gamma} v^\gamma \otimes y^\gamma(s, t) = \underbrace{\iota(T_s)\iota(T_t)\iota(T_s)\dots}_{m_{st} \text{ factors}} - \underbrace{\iota(T_t)\iota(T_s)\iota(T_t)\dots}_{m_{st} \text{ factors}}$$

for some $y^\gamma(s, t) \in \mathbb{Z}\langle e_s, x_{s,\gamma} \rangle$.

Define Ω to be the quotient of $\mathbb{Z}\langle e_s, x_{s,\gamma} \rangle$ modulo the relations

- a.) $e_s^2 = e_s$, $e_s e_t = e_t e_s$,
- b.) $e_s x_{s,\gamma} = x_{s,\gamma}$, $x_{s,\gamma} e_s = 0$,
- c.) $x_{s,\gamma} = x_{s,-\gamma}$ and
- d.) $y^\gamma(s, t) = 0$

for all $s, t \in S$ and all $\gamma \in \Gamma$.

Finally define the element

$$x_s := \sum_{\gamma \in \Gamma} v^\gamma x_{s,\gamma} \in \mathbb{Z}[\Gamma]\Omega.$$

3.5: The definition immediately implies that $T_s \mapsto \iota(T_s)$ defines a homomorphism of $\mathbb{Z}[\Gamma]$ -algebras $\iota : H \rightarrow \mathbb{Z}[\Gamma]\Omega$. In fact this is an embedding as shown in [9, corollary 9]. W -graph algebras have the distinguishing feature that each W -graph (\mathfrak{C}, I, m) with edge weights in k not only defines the structure of a H -module on $k[\Gamma]^{\mathfrak{C}}$ but that it induces a canonical $k\Omega$ -module structure on $k^{\mathfrak{C}}$ via

$$e_s \cdot \mathfrak{z} := \begin{cases} \mathfrak{z} & s \in I(\mathfrak{r}) \\ 0 & \text{otherwise} \end{cases}$$

$$x_s \cdot \mathfrak{z} := \sum_{\mathfrak{r} \in \mathfrak{C}} m_{\mathfrak{r}\mathfrak{z}}^s \mathfrak{r}$$

for all $\mathfrak{z} \in \mathfrak{C}$. Then $k[\Gamma]^{\mathfrak{C} \times \mathfrak{C}}$ is a $k[\Gamma]\Omega$ -module and restriction to a H -module gives back the H -module in the definition.

Conversely if V is a $k\Omega$ -module that has a k -basis \mathfrak{C} w.r.t. which all e_s act as diagonal matrices, then V is obtained from a W -graph (\mathfrak{C}, I, m) in this way. In this way one can interpret Ω -modules as W -graphs up to choice of a basis.

3.6: It is also possible to define a W -graph algebra for an arbitrary weight group Γ , but the definition is more technical. All results of this paper have analogues in the general case.

3.7 Example:

The trivial group is a Coxeter group $(1, \emptyset)$ and its associated W -graph algebra is just \mathbb{Z} .

A cyclic group of order 2 is a Coxeter group $(\{1, s\}, \{s\})$ of rank 1 and its associated W -graph algebra is as a free \mathbb{Z} -module with basis $\{e_s, 1 - e_s\} \cup \{x_{s,\gamma} \mid 0 \leq \gamma < L(s)\}$. The multiplication of the basis elements is completely determined by the relations because $x_{s,\gamma_1} x_{s,\gamma_2} = x_{s,\gamma_1}(e_s x_{s,\gamma_2}) = (x_{s,\gamma_1} e_s) x_{s,\gamma_2} = 0$.

4 Howlett-Yin-Induction

Let M be any $k\Omega$ -module. Then $\mathcal{A} \otimes_k M$ is naturally a $\mathcal{A} \otimes_k k\Omega$ -module and by restriction of scalars it is also a kH -module which we will (somewhat abusing the notation) denote by $\text{Res}_H^\Omega M$.

As a kH_J -module it can be induced to a kH -module. The goal of this subsection is to prove that $\text{Ind}_{H_J}^H \text{Res}_{H_J}^{\Omega_J} M$ not only has a W -graph structure but that this W -graph structure can be chosen functorially in M . The specific construction is a generalisation of such a construction by Howlett and Yin for the special case that M is given by W -graph.

4.1 Preparations

4.1 Proposition:

Let M be a $k\Omega$ -module and $a \in \text{Res}_H^\Omega(M)$ be an arbitrary element. Then the following holds for all $s \in J$: $T_s a = -v_s^{-1} a \iff e_s a = a$

Proof. The forward implication can be seen as follows

$$\begin{aligned}
-v_s^{-1}a &= (-v_s^{-1}e_s + v_s(1 - e_s) + x_s)a \\
\implies 0 &= ((v_s^{-1} + v_s)(1 - e_s) + x_s)a \\
\implies 0 &= ((v_s^{-1} + v_s)(1 - e_s)^2 + \underbrace{(1 - e_s)x_s}_{=0})a \\
&= (v_s^{-1} + v_s)(1 - e_s)a \\
\implies 0 &= (1 - e_s)a
\end{aligned}$$

where we used in the last step that $\text{Res}_H^\Omega(M) = \mathcal{A} \otimes_k M$ as \mathcal{A} -modules to cancel $v_s^{-1} + v_s$. The backward implication is trivial: $T_s a = T_s e_s a = -v_s^{-1} e_s^2 a = -v_s^{-1} a$. \square

We will need the following well-known facts about cosets of parabolic subgroups:

4.2 Lemma and definition:

Let $J \subseteq S$ be any subset and W_J the associated parabolic subgroup. Then the following hold:

- a.) $D_J := \{ x \in W \mid \forall s \in J : l(xs) > l(x) \}$ is a set of representatives for the right cosets of W_J in W . Its elements are exactly the unique elements of minimal length in each coset. They have the property $l(xw) = l(x) + l(w)$ for all $w \in W_J$.
- b.) Deodhar's Lemma (c.f. [1])

For all $w \in D_J$ and all $s \in S$ exactly one of the following cases happens:

- i.) $sw > w$ and $sw \in D_J$
- ii.) $sw > w$ and $sw \notin D_J$. In this case $sw = wt$ for some $t \in J$.
- iii.) $sw < w$. In this case $sw \in D_J$ holds automatically.

Thus for fixed $s \in S$ there is a partition $D_J = D_{J,s}^+ \sqcup D_{J,s}^0 \sqcup D_{J,s}^-$ and similarly for fixed $w \in D_J$ there is a partition $S = D_J^+(w) \sqcup D_J^0(w) \sqcup D_J^-(w)$ where

$$\begin{aligned}
D_{J,s}^+ &:= \{ w \mid sw > w, sw \in D_J \} & D_J^+(w) &:= \{ s \mid sw > w, sw \in D_J \} \\
D_{J,s}^0 &:= \{ w \mid sw > w, sw \notin D_J \} & D_J^0(w) &:= \{ s \mid sw > w, sw \notin D_J \} \\
D_{J,s}^- &:= \{ w \mid sw < w \} & D_J^-(w) &:= \{ s \mid sw < w \}
\end{aligned}$$

If $J \subseteq K \subseteq S$ is another subset, then furthermore

- c.) $D_J^K := D_J \cap W_K$ is the set of distinguished right coset representatives for W_J in W_K and $D_K^S \times D_J^K \rightarrow D_J^S, (x, y) \mapsto xy$ is a length-preserving bijection.
- d.) If $x \in D_K^S$ and $y \in D_J^K$, then

$$\begin{aligned}
D_J^+(xy) &= \{ s \in D_K^0(x) \mid s^x \in D_J^+(y) \} \cup D_K^+(x) \\
D_J^0(xy) &= \{ s \in D_K^0(x) \mid s^x \in D_J^0(y) \} \\
D_J^-(xy) &= \{ s \in D_K^0(x) \mid s^x \in D_J^-(y) \} \cup D_K^-(x)
\end{aligned}$$

Proof. See [6] for example. \square

To apply the previous observations about canonical modules we need a semilinear map on our modules.

4.3 Lemma and definition:

If M is any $k\Omega$ -module, then we will fix the notation ι for the canonical $\bar{}$ -semilinear map $a \otimes m \mapsto \bar{a} \otimes m$ on $\mathcal{A} \otimes_k M$.

Then the following hold:

- a.) ι is ring automorphism of $\mathcal{A} \otimes_k k\Omega$ with $\iota(T_s) = T_s^{-1} = T_s - (v_s - v_s^{-1})$. In particular ι restricts to a $\bar{}$ -semilinear involution of kH . Furthermore $\iota(ax) = \iota(a)\iota(x)$ for all $a \in \mathcal{A} \otimes \Omega$, $x \in \mathcal{A} \otimes M$.

Now let M be a $k\Omega_J$ -module and $V := \text{Res}_{H_J}^{\Omega_J}(M)$ its associated kH_J -module.

- b.) $\iota(h \otimes x) := \iota(h) \otimes \iota(x)$ is a well-defined $\bar{}$ -semilinear involution on $\text{Ind}_{H_J}^H(V) = H \otimes_{H_J} V$.
- c.) The k -submodules $V_w := \{ T_w \otimes m \mid m \in M \} \subseteq \text{Ind}_{H_J}^H(V)$ for $w \in D_J$ constitute a D_J -graded shadow on $\text{Ind}_{H_J}^H(V)$ where D_J is endowed with the Bruhat-Chevalley-order.
- d.) The maps $\rho_{xz} : V_z \rightarrow \mathcal{A}V_x$ and $\pi_{xz} : V_z \rightarrow \mathcal{A}V_x$ in theorem 2.7 are of the form

$$\rho_{xz}(T_z \otimes m) = T_x \otimes r_{xz}m \quad \text{and} \quad \pi_{xz}(T_z \otimes m) = T_x \otimes p_{xz}m$$

for elements $r_{xz} \in \mathbb{Z}[\Gamma]\Omega_J$, $p_{xz} \in \mathbb{Z}[\Gamma_{\geq 0}]\Omega_J$ that are independent of M .

Proof. a. follows directly from the definition.

b. For all $a \in H_J$ one has $\iota(ha) \otimes \iota(x) = \iota(h)\iota(a) \otimes \iota(x) = \iota(h) \otimes \iota(a)\iota(x) = \iota(h) \otimes \iota(ax)$ which proves the well-definedness of $h \otimes x \mapsto \iota(h) \otimes \iota(x)$.

c. Note that

$$\iota(T_z \otimes m) = \sum_{w \in W} R_{wz} T_w \otimes m = \sum_{x \in D_J} T_x \otimes \underbrace{\sum_{y \in W_J} R_{xy,z} T_y}_{=: r_{x,z}} m$$

Now note that $R_{xy,z} \neq 0$ implies $xy \leq z$ so that $x \leq xy \leq z$ and thus the summation only runs over x with $x \leq z$. If furthermore $x = z$ then $xy \leq z$ can only be true if $y = 1$. But we know $R_{zz} = 1$. Therefore (V_w) really is a D_J -graded shadow of V .

d. The claimed property for ρ_{xz} follows from the equation above. The analogous property for π_{xz} follows from the recursive construction of the π_{xz} in theorem 2.7. \square

4.4 Lemma and definition:

There is a unique family $\mu_{x,z}^s \in \mathbb{Z}[\Gamma]\Omega_J$ for $x, z \in D_J, s \in S$ such that the following properties hold

a.) $\mu_{x,z}^s = 0$ unless $x < z$, $z \in D_{J,s}^+ \cup D_{J,s}^0$, and $x \in D_{J,s}^0 \cup D_{J,s}^-$ hold.

b.) $\overline{\mu_{x,z}^s} = \mu_{x,z}^s$

c.) If $z \in D_{J,s}^+ \cup D_{J,s}^0$ and $x \in D_{J,s}^0 \cup D_{J,s}^-$, then

$$\mu_{x,z}^s + R + \sum_{x < y < z} p_{x,y} \mu_{y,z}^s \in \mathbb{Z}[\Gamma_{>0}] \Omega_J$$

where

$$R = \begin{cases} -C_{s^x} p_{x,z} & z \in D_{J,s}^+, x \in D_{J,s}^0 \\ v_s^{-1} p_{x,z} & z \in D_{J,s}^+, x \in D_{J,s}^- \\ p_{x,z} C_{s^z} - C_{s^x} p_{x,z} & z \in D_{J,s}^0, x \in D_{J,s}^0 \\ p_{x,z} C_{s^z} + v_s^{-1} p_{x,z} & z \in D_{J,s}^0, x \in D_{J,s}^- \end{cases}$$

These elements satisfy:

d.) $v_s \mu_{x,z}^s \in \mathbb{Z}[\Gamma_{>0}] \Omega_J$.

Proof. $\mu_{x,z}^s$ is well-defined as zero if the conditions of a. do not hold. If they hold, then b. and c. imply a recursive algorithm to compute $\mu_{x,z}^s$ for $x < z$.

Consider the poset $\{(x, z) \in D_J \times D_J \mid x \leq z\}$ with the order $(x, z) \sqsubset (x', z') : \iff z < z' \vee (z = z' \wedge x > x')$ and note that this is a well-founded poset since intervals in the Bruhat-Chevalley order are finite so that no infinite descending chain can exist. The recursion happens along this poset. If $\mu_{x,z}^s$ is known for all $(x, z) \sqsubset (x', z')$ then c. determines the nonpositive part of $\mu_{x',z'}^s$ and by the symmetry condition $\mu_{x',z'}^s$ is completely determined.

The last property of $\mu_{x,z}^s$ follows from by induction from this recursive construction. Note that $v_s R \in \mathbb{Z}[\Gamma_{>0}] \Omega_J$ in all four cases because $p_{x,z} \in \mathbb{Z}[\Gamma_{>0}] \Omega_J$ for all $x < z$. Assuming that $v_s \mu_{y,z}^s \in \mathbb{Z}[\Gamma_{>0}] \Omega_J$ already holds for all $(y, z) \sqsubset (x, z)$ we find that

$$v_s \mu_{x,z}^s \equiv -v_s R - \sum_{x < y < z} p_{x,y} v_s \mu_{y,z}^s \equiv 0 \pmod{\mathbb{Z}[\Gamma_{>0}] \Omega_J}$$

holds. □

4.2 The main theorem

4.5 Theorem:

Let M be a $k\Omega_J$ -module and $\text{HY}_J^S(M)$ the k -module $\bigoplus_{w \in D_J} M$. Denote elements of the w -component of $\text{HY}_J^S(M)$ as $w|m$. Further write $\mu_{x,z}^s = \sum_{-L(s) < \gamma < L(s)} \mu_{x,z}^{s,\gamma} \cdot v^\gamma$ with $\mu_{x,z}^{s,\gamma} \in \Omega_J$.

With this notation $\text{HY}_J^S(M)$ becomes a $k\Omega$ -module via

$$e_s \cdot z|m := \begin{cases} 0 & z \in D_{J,s}^+ \\ z|e_s z m & z \in D_{J,s}^0 \\ z|m & z \in D_{J,s}^- \end{cases}$$

$$x_{s,\gamma} \cdot z|m := \begin{cases} \sum_{x < z} x|\mu_{x,z}^{s,\gamma} m + sz|m & z \in D_{J,s}^+, \gamma = 0 \\ \sum_{x < z} x|\mu_{x,z}^{s,\gamma} m & z \in D_{J,s}^+, \gamma \neq 0 \\ \sum_{x < z} x|\mu_{x,z}^{s,\gamma} m + z|x_{sz,\gamma} m & z \in D_{J,s}^0 \\ 0 & z \in D_{J,s}^- \end{cases}$$

and the canonicalisation $c_M : \text{Res}_H^\Omega \text{HY}_J^S(M) \rightarrow \text{Ind}_{H_J}^H \text{Res}_{H_J}^{\Omega_J}(M), z|m \mapsto \sum_y T_y \otimes p_{yz} m$ is kH -linear.

Proof. Denote these k -linear maps $\text{HY}_J^S(M) \rightarrow \text{HY}_J^S(M)$ by $\omega(e_s)$ and $\omega(x_{s,\gamma})$ and extend this \mathcal{A} -linearly to $\omega : \text{span}_{\mathcal{A}} \{1, e_s, x_{s,\gamma} \mid s \in S\} \rightarrow \mathcal{A}\text{End}(\text{HY}_J^S(M))$.

We need to show that ω satisfies all the relations of Ω , that is

- a.) $e_s^2 = e_s, \quad e_s e_t = e_t e_s,$
- b.) $e_s x_{s,\gamma} = x_{s,\gamma}, \quad x_{s,\gamma} e_s = 0,$
- c.) $x_{s,\gamma} = x_{s,-\gamma}$ and
- d.) $\underbrace{T_s T_t T_s \dots}_{m_{st} \text{ factors}} = \underbrace{T_t T_s T_t \dots}_{m_{st} \text{ factors}}$ as an equation in $\mathcal{A} \otimes_k \Omega$ where $m_{st} := \text{ord}(st)$.

The equations $\omega(e_s)^2 = \omega(e_s)$, $\omega(e_s)\omega(e_t) = \omega(e_t)\omega(e_s)$ and $\omega(x_{s,\gamma}) = \omega(x_{s,-\gamma})$ follow directly from the definition and the properties of μ .

To prove that $\omega(T_s) \in \mathcal{A}\text{End}(\text{HY}_J^S(M))$ satisfies the braid relations, we use the \mathcal{A} -linear bijection c and show $c(\omega(T_s)z|m) = T_s c(z|m)$ for all $z \in D_J$ and all $m \in M$. Since $\text{Ind}_{H_J}^H \text{Res}_{H_J}^{\Omega_J}(M)$ is a kH -modul, the braid relations hold on the right hand side and thus also on the left hand side. Because of the equality $C_s = T_s - v_s$ this is equivalent to showing $c(\omega(C_s)w|m) = C_s c(w|m)$. We compare these two elements of $\text{Ind}_{H_J}^H \text{Res}_{H_J}^{\Omega_J} M$:

One the left hand side we find:

$$\begin{aligned}
c(\omega(C_s)z|m) &= c((-v_s + v_s^{-1})e_s + x_s) \cdot z|m) \\
&= \begin{cases} c(sz|m + \sum_{y < z} y|\mu_{y,z}^s m) & z \in D_{J,s}^+ \\ c(z|(-(v_s + v_s^{-1})e_{s^z} + x_{s^z})m + \sum_{y < z} y|\mu_{y,z}^s m) & z \in D_{J,s}^0 \\ -(v_s^{-1} + v_s)c(z|m) & z \in D_{J,s}^- \end{cases} \\
&= \begin{cases} c(v_s z|m + sz|m + \sum_{y < z} y|\mu_{y,z}^s m) & z \in D_{J,s}^+ \\ c(z|C_{s^z}m + \sum_{y < z} y|\mu_{y,z}^s m) & z \in D_{J,s}^0 \\ -(v_s + v_s^{-1})c(z|m) & z \in D_{J,s}^- \end{cases} \\
&= \begin{cases} \sum_{x \in D_{J,s}} T_x \otimes \left(p_{x,s^z}m + \sum_{x \leq y < z} p_{x,y}\mu_{y,z}^s m \right) & z \in D_{J,s}^+ \\ \sum_{x \in D_{J,s}} T_x \otimes \left(p_{x,z}C_{s^z}m + \sum_{x \leq y < z} p_{x,y}\mu_{y,z}^s m \right) & z \in D_{J,s}^0 \\ \sum_{x \in D_{J,s}} T_x \otimes (-v_s - v_s^{-1})p_{x,z}m & z \in D_{J,s}^- \end{cases}
\end{aligned}$$

On right hand side we find:

$$\begin{aligned}
C_s c(z|m) &= \sum_{x \in D_{J,s}} T_s T_x \otimes p_{x,z}m + T_x \otimes (-v_s)p_{x,z}m \\
&= \sum_{x \in D_{J,s}^+} T_{sx} \otimes p_{x,z}m + T_x \otimes (-v_s)p_{x,z}m \\
&\quad + \sum_{x \in D_{J,s}^0} T_x \otimes T_{s^x}p_{x,z}m + T_x \otimes (-v_s)p_{x,z}m \\
&\quad + \sum_{x \in D_{J,s}^-} (T_{sx} + (v_s - v_s^{-1})T_x) \otimes p_{x,z}m + T_x \otimes (-v_s)p_{x,z}m \\
&= \sum_{x \in D_{J,s}^-} T_x \otimes p_{sx,z}m + \sum_{x \in D_{J,s}^+} T_x \otimes (-v_s)p_{x,z}m \\
&\quad + \sum_{x \in D_{J,s}^0} T_x \otimes (T_{s^x} - v_s)p_{x,z}m \\
&\quad + \sum_{x \in D_{J,s}^+} T_x \otimes p_{sx,z}m + \sum_{x \in D_{J,s}^-} T_x \otimes (-v_s^{-1})p_{x,z}m \\
&= \sum_{x \in D_{J,s}^+} T_x \otimes (p_{sx,z} - v_s p_{x,z})m \\
&\quad + \sum_{x \in D_{J,s}^0} T_x \otimes C_{s^x}p_{x,z}m
\end{aligned}$$

$$+ \sum_{x \in D_{J,s}^-} T_x \otimes (p_{sx,z} - v_s^{-1} p_{x,z}) m$$

Comparing the $T_x \otimes M$ components we find an equation of elements of $\mathbb{Z}[\Gamma]\Omega_J$ that needs to be satisfied. More specifically it is the equation 4.6.a below.

Similarly the equations

$$\omega(e_s)\omega(x_s) = \omega(x_s) \quad \text{and} \quad \omega(x_s)\omega(e_s) = 0$$

translate into equations of elements of $\mathbb{Z}[\Gamma]\Omega_J$, the two equations 4.6.c and 4.6.d below.

The remainder of this subsection is devoted to proving these equations:

4.6 Lemma:

The elements p_{xz}, μ_{xz}^s of $\mathbb{Z}[\Gamma]\Omega_J$ satisfy the following equations:

a.) For all $z \in D_J$ and all $x \in D_J$

$$\left. \begin{array}{ll} x \in D_{J,s}^+ & p_{sx,z} - v_s p_{x,z} \\ x \in D_{J,s}^0 & C_{s^x} p_{x,z} \\ x \in D_{J,s}^- & p_{sx,z} - v_s^{-1} p_{x,z} \end{array} \right\} = \left\{ \begin{array}{ll} p_{x,sz} + \sum_{x \leq y < z} p_{x,y} \mu_{y,z}^s & z \in D_{J,s}^+ \\ p_{x,z} C_{s^z} + \sum_{x \leq y < z} p_{x,y} \mu_{y,z}^s & z \in D_{J,s}^0 \\ -(v_s + v_s^{-1}) p_{x,z} & z \in D_{J,s}^- \end{array} \right.$$

b.) For all $z \in D_{J,s}^0$ and all $x \in D_J$:

$$p_{x,z} e_{s^z} = \begin{cases} -v_s p_{sx,z} e_{s^z} & x \in D_{J,s}^+ \\ e_{s^x} p_{x,z} e_{s^z} & x \in D_{J,s}^0 \\ -v_s^{-1} p_{sx,z} e_{s^z} & x \in D_{J,s}^- \end{cases}$$

c.) For all $z \in D_J$ and all $x \in D_{J,s}^0$: $e_{s^x} \mu_{x,z}^s = \mu_{x,z}^s$

d.) For all $z \in D_{J,s}^0$ and all $x \in D_J$: $\mu_{x,z}^s e_{s^z} = 0$

We will prove these four equations simultaneously with a double induction. We will induct over $l(z)$ and assume that all four equations hold for all pairs (x', z') with $l(z') < l(z)$. For any fixed z we will induct over $l(z) - l(x)$. Observe that all equations are trivially true if $l(x) > l(z) + 1$ because all occurring p and μ are zero. We will therefore assume that the equations also hold for all pairs (x', z) with $l(x') > l(x)$.

Proof of 4.6.a. We denote with f_{xz} the difference between the right hand side and the left hand side of the equation. Then by the above considerations:

$$c(\omega(C_s)z|m) - C_s c(z|m) = \sum_x T_x \otimes f_{xz} m$$

We will show $f_{xz} \in \mathbb{Z}[\Gamma_{>0}]\Omega_J$ and conclude $f_{xz} = 0$ using lemma 2.6. Note that both $c(\omega(C_s)z|m)$ as well as $C_s c(z|m)$ are ι -invariant elements of $\text{Ind}_{H_J}^H \text{Res}_{H_J}^{\Omega_J} M$ because c is $\hat{\mathcal{A}}$ -linear and $\iota(C_s) = C_s$.

Case 1: $z \in D_{J,s}^+$.

Case 1.1.+1.2: $x \in D_{J,s}^0 \cup x \in D_{J,s}^-$

In both cases $f_{xz} \in \mathbb{Z}[\Gamma_{>0}]\Omega_J$ by definition of μ .

Case 1.3.: $x \in D_{J,s}^+$.

Observe that $p_{sx,z} = 1 \iff sx = z \iff x = sz \iff p_{x,sz} = 1$ so that $p_{sx,z} - p_{x,sz}$ is always an element of $\mathbb{Z}[\Gamma_{>0}]\Omega_J$. Thus

$$\begin{aligned} f_{xz} &= \underbrace{p_{z,sx} - p_{sx,z}}_{\in \mathbb{Z}[\Gamma_{>0}]\Omega_J} + \underbrace{v_s p_{x,z}}_{\in \mathbb{Z}[\Gamma_{>0}]} + \sum_y p_{xy} \mu_{yz}^s \\ &\equiv \sum_y p_{xy} \mu_{yz}^s \pmod{\mathbb{Z}[\Gamma_{>0}]\Omega_J} \\ &= \sum_{\substack{y \in D_{J,s}^+ \\ x \leq y < z}} p_{xy} \underbrace{\mu_{yz}^s}_{=0} + \sum_{\substack{y \in D_{J,s}^0 \\ x \leq y < z}} p_{xy} \mu_{yz}^s + \sum_{\substack{y \in D_{J,s}^- \\ x \leq y < z}} p_{xy} \mu_{yz}^s \end{aligned}$$

Now since $x \in D_{J,s}^+$ we cannot have $x = y$ in the both the second and third sum so that we can use 4.6.a for (x, y) and 4.6.c for (y, z) in the second sum as well as 4.6.a for (x, y) in the third sum so that we obtain

$$\begin{aligned} &= \sum_{\substack{y \in D_{J,s}^0 \\ x \leq y < z}} (-v_s^{-1})(p_{x,y} C_{s^y} - p_{sx,y} + \sum_{x \leq y' < y} p_{x,y'} \mu_{y',y}^s) (e_{s^y} \mu_{yz}^s) + \sum_{\substack{y \in D_{J,s}^- \\ \in \mathbb{Z}[\Gamma_{\geq 0}]} } \underbrace{(p_{sx,y} (-v_s))}_{\in \mathbb{Z}[\Gamma_{>0}]} \mu_{y,z}^s \\ &\equiv \sum_{\substack{y \in D_{J,s}^0 \\ x \leq y < z}} (-v_s^{-1})(p_{x,y} C_{s^y} - p_{sx,y} + \sum_{y'} p_{x,y'} \mu_{y',y}^s) e_{s^y} \mu_{yz}^s \pmod{\mathbb{Z}[\Gamma_{>0}]\Omega_J} \end{aligned}$$

Now $y < z$ so that we can use 4.6.d for (y', y) and obtain

$$\begin{aligned} &= \sum_{\substack{y \in D_{J,s}^0 \\ x \leq y < z}} (-v_s^{-1})(p_{x,y} C_{s^y} - p_{sx,y} + \sum_{y'} p_{x,y'} \underbrace{\mu_{y',y}^s}_{=0}) e_{s^y} \mu_{yz}^s \\ &= \sum_{\substack{y \in D_{J,s}^0 \\ x \leq y < z}} (-v_s^{-1})(p_{x,y} C_{s^y} e_{s^y} - p_{sx,y} e_{s^y}) \mu_{y,z}^s \\ &= \sum_{\substack{y \in D_{J,s}^0 \\ x \leq y < z}} (-v_s^{-1})(p_{x,y} (-v_s - v_s^{-1}) e_{s^y} - p_{sx,y} e_{s^y}) \mu_{y,z}^s \end{aligned}$$

Applying 4.6.b to (x, y) we find

$$= \sum_{y \in D_{J,s}^0} (-v_s^{-1})(-v_s p_{sx,y} e_{s^y} (-v_s - v_s^{-1}) - p_{sx,y} e_{s^y}) \mu_{y,z}^s$$

$$\begin{aligned}
&= \sum_{y \in D_{J,s}^0} (-v_s^{-1})(v_s^2 p_{sx,y} e_{sy}) \mu_{y,z}^s \\
&= \sum_{y \in D_{J,s}^0} - \underbrace{p_{sx,y}}_{\in \mathbb{Z}[\Gamma_{\geq 0}]\Omega_J} e_{sy} \underbrace{v_s \mu_{y,z}^s}_{\in \mathbb{Z}[\Gamma_{>0}]\Omega_J}
\end{aligned}$$

Case 2: $z \in D_{J,s}^0$.

Case 2.1.+2.2: $x \in D_{J,s}^0 \cup x \in D_{J,s}^-$

In both cases $f_{xz} \in \mathbb{Z}[\Gamma_{>0}]\Omega_J$ by definition of μ .

Case 2.3. $x \in D_{J,s}^+$.

In this case $sx \neq z$ so that $p_{sx,z} \in \mathbb{Z}[\Gamma_{>0}]\Omega_J$. Therefore:

$$\begin{aligned}
f_{xz} &= p_{x,z} C_{sz} + \sum_{y < z} p_{x,y} \mu_{y,z}^s - p_{sx,z} + v_s p_{x,z} \\
&\equiv p_{x,z} T_{sz} + \sum_{y < z} p_{x,y} \mu_{y,z}^s \pmod{\mathbb{Z}[\Gamma_{>0}]\Omega_J}
\end{aligned}$$

Because $x' := sx$ satisfies $x' > x$ and $x' \in D_{J,s}^-$ we can use the induction hypothesis for (x', z) and (x', y) so that

$$\begin{aligned}
&= p_{sx',z} T_{sz} + \sum_{y < z} p_{sx',y} \mu_{y,z}^s \\
&= \underbrace{(p_{x',z} C_{sz} + v_s^{-1} p_{x',z})}_{= p_{x',z} T_{sz}^{-1}} + \sum_{y'} p_{x',y'} \mu_{y',z}^s T_{sz} + \sum_{y \in D_{J,s}^+} p_{x,y} \underbrace{\mu_{y,z}^s}_{=0} \\
&\quad + \sum_{y \in D_{J,s}^0} \underbrace{(p_{x',y} C_{sy} + v_s^{-1} p_{x',y})}_{= p_{x',y} T_{sy}^{-1}} + \sum_{y'} p_{x',y'} \mu_{y',y}^s \mu_{y,z}^s \\
&\quad + \sum_{y \in D_{J,s}^-} (-(v_s + v_s^{-1}) p_{x',y} + v_s^{-1} p_{x',y}) \mu_{y,z}^s \\
&= \underbrace{p_{x',z}}_{\in \mathbb{Z}[\Gamma_{>0}]\Omega_J} + \sum_{x' \leq y' < z} p_{x',y'} \mu_{y',z}^s T_{sz} \\
&\quad + \sum_{\substack{y \in D_{J,s}^0 \\ x' \leq y}} (p_{x',y} T_{sy}^{-1} + \sum_{x' \leq y' < y} p_{x',y'} \mu_{y',y}^s) \mu_{y,z}^s \\
&\quad + \sum_{y \in D_{J,s}^-} - \underbrace{p_{x',y}}_{\in \mathbb{Z}[\Gamma_{\geq 0}]\Omega_J} \cdot \underbrace{v_s \mu_{y,z}^s}_{\in \mathbb{Z}[\Gamma_{>0}]\Omega_J} \\
&\equiv \sum_{x' \leq y' < z} p_{x',y'} \mu_{y',z}^s (-v_s^{-1} e_{sz} + v_s(1 - e_{sz}) + x_{sz}) \\
&\quad + \sum_{\substack{y \in D_{J,s}^0 \\ x' \leq y}} p_{x',y} T_{sy}^{-1} \mu_{y,z}^s + \sum_{\substack{y \in D_{J,s}^0, y' \in D_J \\ x' \leq y' < y < z}} p_{x',y'} \mu_{y',y}^s \mu_{y,z}^s \pmod{\mathbb{Z}[\Gamma_{>0}]\Omega_J}
\end{aligned}$$

Because $x < x'$ we can use lemma 4.6.d for (y', z) in the first sum, 4.6.c for (y, z) in the second and third sum as well as 4.6.d for (y', y) in the third sum to obtain

$$\begin{aligned}
&= \sum_{\substack{x' \leq y' < z \\ \in \mathbb{Z}[\Gamma_{\geq 0}]\Omega_J}} \underbrace{p_{x', y'}}_{\in \mathbb{Z}[\Gamma_{\geq 0}]\Omega_J} \underbrace{\mu_{y', z}^s v_s}_{\in \mathbb{Z}[\Gamma_{> 0}]\Omega_J} (1 - e_{sz}) \\
&+ \sum_{\substack{y \in D_{J, s}^0 \\ x' \leq y}} p_{x', y} \underbrace{T_{sy}^{-1} e_{sy}}_{= -v_s e_{sy}} \mu_{y, z}^s + \sum_{\substack{y \in D_{J, s}^0, y \in D_J \\ x' \leq y' < y < z}} p_{x', y'} \underbrace{\mu_{y', y}^s e_{sy}}_{= 0} \mu_{y, z}^s \\
&\equiv \sum_{\substack{y \in D_{J, s}^0 \\ x' \leq y}} \underbrace{p_{x', y}}_{\in \mathbb{Z}[\Gamma_{\geq 0}]}} \underbrace{(-e_{sy})}_{\in \mathbb{Z}[\Gamma_{> 0}]}} \underbrace{v_s \mu_{y, z}^s}_{\in \mathbb{Z}[\Gamma_{> 0}]}} \mod \mathbb{Z}[\Gamma_{> 0}]\Omega_J \\
&\equiv 0 \mod \mathbb{Z}[\Gamma_{> 0}]\Omega_J
\end{aligned}$$

which is what we wanted to prove.

Case 3: $z \in D_{J, s}^-$.

Case 3.1.: $x \in D_{J, s}^-$.

In this case $sx \in D_{J, s}^+$ so that $sx \neq z$ and thus $p_{sx, z} \in \mathbb{Z}[\Gamma_{> 0}]\Omega_J$. We infer

$$f_{xz} = p_{sx, z} - v_s^{-1} p_{xz} + v_s p_{xz} + v_s^{-1} p_{xz} = p_{sx, z} + v_s p_{x, z} \in \mathbb{Z}[\Gamma_{> 0}]\Omega_J$$

Case 3.2.: $x \in D_{J, s}^0$.

In this case the equation is equivalent to $(1 - e_{sx})p_{x, z} = 0$ by proposition 4.1. If $x \not\leq z$ then this is vacuously true because $p_{xz} = 0$.

Since $z \in D_{J, s}^-$ we can write $z = sz'$ for some $z' \in D_{J, s}^+$ with $z' < z$. We can apply 4.6.a to (x, z') and find:

$$\begin{aligned}
(1 - e_{sx})p_{x, z} &= (1 - e_{sx})p_{x, sz'} \\
&= \underbrace{(1 - e_{sx})(C_{sx})}_{= C_{sx}} p_{x, z'} - \sum_y p_{x, y} \mu_{y, z'}^s \\
&= - \sum_{y \in D_{J, s}^+} (1 - e_{sx})p_{x, y} \underbrace{\mu_{y, z'}^s}_{= 0} \\
&- \sum_{\substack{y \in D_{J, s}^0 \\ x \leq y < z'}} (1 - e_{sx})p_{x, y} \mu_{y, z'}^s - \sum_{\substack{y \in D_{J, s}^- \\ x \leq y < z'}} (1 - e_{sx})p_{x, y} \mu_{y, z'}^s
\end{aligned}$$

Because $z' < z$ we can apply 4.6.c to (y, z') in the second sum and 4.6.a to (x, y) (in the equivalent formulation $(1 - e_{sx})p_{x, y} = 0$) in the third sum so that we obtain

$$= - \sum_{\substack{y \in D_{J, s}^0 \\ x \leq y < z'}} (1 - e_{sx})p_{x, y} e_{sy} \mu_{y, z'}^s$$

Using 4.6.b for (x, y) all the summands vanish.

Case 3.3.: $x \in D_{J,s}^+$.

In this case $x' := sx$ satisfies $x' > x$ and $x' \in D_{J,s}^-$ so that we find

$$\begin{aligned} f_{xz} &= p_{sx,z} - v_s p_{x,z} + v_s p_{x,z} + v_s^{-1} p_{x,z} \\ &= p_{x',z} + v_s^{-1} p_{sx',z} \end{aligned}$$

which – using the induction hypothesis for (x', z) – equals

$$\begin{aligned} &= p_{x',z} + v_s^{-1} (-v_s p_{x',z}) \\ &= 0 \end{aligned}$$

This concludes the proof of 4.6.a. □

Since we have proven equation 4.6.a we can use it to prove the other equations:

Proof of 4.6.b and 4.6.d. If $x \in D_{J,s}^\pm$ we can multiply the equation 4.6.a with e_{s^z} from the right:

$$\begin{aligned} (p_{sx,z} - v_s^{-1} p_{x,z}) e_{s^z} &= p_{x,z} C_{s^z} e_{s^z} + \mu_{xz}^s e_{s^z} + \sum_{x < y < z} p_{xy} \mu_{yz}^s e_{s^z} \\ &= p_{x,z} (-v_s - v_s^{-1}) e_{s^z} + \mu_{xz}^s e_{s^z} + \sum_{x < y < z} p_{xy} \mu_{yz}^s e_{s^z} \end{aligned}$$

Now we can apply 4.6.d to (y, z) in the sum and obtain:

$$\begin{aligned} &= p_{x,z} (-v_s - v_s^{-1}) e_{s^z} + \mu_{xz}^s e_{s^z} \\ \implies (p_{sx,z} + v_s^{-1} p_{x,z}) e_{s^z} &= \mu_{xz}^s e_{s^z} \end{aligned}$$

If $x \in D_{J,s}^+$, then $\mu_{xz}^s = 0$ so that $p_{sx,z} = -v_s^{-1} p_{x,z}$ and $\mu_{xz}^s e_{s^z} = 0$. If $x \in D_{J,s}^-$, then the left hand is contained in $\mathbb{Z}[\Gamma_{>0}] \Omega_J$ because $sx, x \neq z$ while the right hand side is $\overline{}$ -invariant so that both sides equal to zero.

Now consider the case $x \in D_{J,s}^0$. Then the following holds by equation 4.6.a for (x, z) :

$$\begin{aligned} (v_s + v_s^{-1}) p_{xz} e_{s^z} &= -p_{xz} C_{s^z} e_{s^z} \\ &= -C_{s^x} p_{x,z} e_{s^z} + \mu_{xz} e_{s^z} + \sum_{x < y < z} p_{xy} \mu_{yz}^s e_{s^z} \end{aligned}$$

In the sum we can use 4.6.d for (y, z) and find:

$$\begin{aligned} (v_s + v_s^{-1}) p_{xz} e_{s^z} &= -C_{s^x} p_{x,z} e_{s^z} + \mu_{xz} e_{s^z} \\ \implies \mu_{xz}^s e_{s^z} &= (C_{s^x} + v_s + v_s^{-1}) p_{xz} e_{s^z} \\ &= ((v_s + v_s^{-1})(1 - e_s) + x_s) p_{xz} e_{s^z} \\ &= ((v_s + v_s^{-1}) + x_s)(1 - e_s) p_{xz} e_{s^z} \end{aligned}$$

Now consider the exponents that occur at both sides of the equation: On the left hand side all exponents are $< L(s)$. On the right hand side this means $v_s(1 - e_s)p_{xz}e_{sz} = 0$ because $p_{xz} \in \mathbb{Z}[\Gamma_{\geq 0}]\Omega_J$. Therefore $(1 - e_s)p_{xz}e_{sz} = 0$ and $\mu_{yz}^s e_{sz} = 0$. This concludes the proof of [4.6.b](#) and [4.6.d](#). \square

Proof of lemma [4.6.c](#). Wlog we assume $x < z$ and $z \in D_{J,s}^+ \cup D_{J,s}^0$ since otherwise $\mu_{xz}^s = 0$. Because $\overline{\mu_{xz}^s} = \mu_{xz}^s$ we only need to prove $(1 - e_{s^x})\mu_{xz}^s \equiv 0 \pmod{\mathbb{Z}[\Gamma_{>0}]\Omega_J}$. Lemma [4.6.a](#) for (x, z) implies

$$\begin{aligned} 0 &= (1 - e_{s^x})C_{s^x}p_{xz} \\ &= (1 - e_{s^x})p_{xx}\mu_{xz}^s + \left\{ \begin{array}{ll} (1 - e_{s^x})p_{x,sz} & z \in D_{J,s}^+ \\ (1 - e_{s^x})p_{xz}C_{sz} & z \in D_{J,s}^0 \end{array} \right\} + \sum_{x < y < z} (1 - e_{s^x})p_{xy}\mu_{yz}^s \end{aligned}$$

Because $x \in D_{J,s}^0$, $x = sz$ cannot hold in the first case so that $p_{x,sz} \in \mathbb{Z}[\Gamma_{>0}]\Omega_J$. In the second case we use $C_{sz} = e_{sz}C_{sz}$ and the now proven [4.6.b](#). We obtain

$$\equiv (1 - e_{s^x})\mu_{xz}^s + \sum_{\substack{y \in D_{J,s}^0 \\ x < y < z}} (1 - e_{s^x})p_{xy}\mu_{yz}^s + \sum_{\substack{y \in D_{J,s}^- \\ x < y < z}} (1 - e_{s^x})p_{xy}\mu_{yz}^s \pmod{\mathbb{Z}[\Gamma_{>0}]\Omega_J}$$

Note that $(1 - e_{s^x})p_{xy} = 0$ for $y \in D_{J,s}^-$ by [4.6.a](#) applied to (x, y) and $\mu_{y,z}^s = e_{sy}\mu_{y,z}^s$ for $y \in D_{J,s}^0$ by [4.6.c](#). Using [4.6.b](#) we find

$$\begin{aligned} &= (1 - e_{s^x})\mu_{xz}^s + \sum_{\substack{y \in D_{J,s}^0 \\ x < y < z}} \underbrace{(1 - e_{s^x})p_{xy}e_{sy}}_{=0} \mu_{yz}^s \\ &= (1 - e_{s^x})\mu_{xz}^s \end{aligned}$$

This concludes the proof of Lemma [4.6.c](#). \square

We have therefore finally completed the inductive proof of [4.6](#) and therefore the proof of the main theorem. \square

4.3 Applications

4.7: Note that the equation [4.6.a](#) and definition of the μ lead to the following recursive algorithm to compute $p_{x,z}$ and $\mu_{x,z}^s$ for all $x, z \in D_J$ and all $s \in S$. The recursion is along the (well-founded!) order $(x, z) \sqsubset (x', z') : \iff z < z' \vee (z = z' \wedge x > x')$ on $\{(x, z) \in D_J \times D_J \mid x \leq z\}$.

1. If $x = z$, then $p_{x,z} = 1$.
2. If $x < z$, then choose any $t \in S$ with $tz < z$ and consider the following cases:
 - 2.1. If $t \in D_J^+(x)$, then $p_{x,z} = -v_t p_{tx,z}$.
 - 2.2. If $t \in D_J^0(x)$, then $p_{x,z} = C_{t^x} p_{x,tz} - \sum_{y < tz} p_{x,y} \mu_{y,tz}^t$

- 2.3. If $t \in D_J^-(x)$, then $p_{x,z} = p_{tx,tz} - v_t^{-1} p_{x,tz} - \sum_{y < tz} p_{x,y} \mu_{y,tz}^t$
3. If $s \in D_J^+(x)$ or $s \in D_J^-(z)$, then $\mu_{x,z}^s = 0$.
4. Otherwise compute $\alpha := -R - \sum_{x < y < z} p_{x,y} \mu_{y,z}^s$, where R is defined as in 4.4. Write $\alpha = \alpha_- + \alpha_0 + \alpha_+$ where $\alpha_- \in \mathbb{Z}[\Gamma_{<0}] \Omega_J$, $\alpha_0 \in \Omega_J$, $\alpha_+ \in \mathbb{Z}[\Gamma_{>0}] \Omega_J$. Then $\mu_{x,z}^s = \alpha_- + \alpha_0 + \overline{\alpha_+}$.

4.8 Example:

Starting with $J = \emptyset$ and the regular module $\Omega_\emptyset = \mathbb{Z}$ we obtain the special case $\text{HY}_\emptyset^S(\mathbb{Z}) =: KL^S$. As a H -module this is isomorphic to $\text{Ind}_{H_\emptyset}^H(H_\emptyset) = H$ and the basis $\{z|1 \mid z \in W\}$ is identified with the Kazhdan-Lusztig basis $\{C_z \mid z \in W\}$ via the canonicalisation map.

Thus we recover Kazhdan and Lusztig's result (c.f. [12, 1.3]) that the regular H -module is induced by a W -graph. The elements $\mu_{x,y}^s \in \mathbb{Z}[\Gamma] \Omega_\emptyset = \mathbb{Z}[\Gamma]$ equal the μ -values defined in [12] and [13] (in the case of unequal parameters) up to a sign. The elements $p_{x,y} \in \mathbb{Z}[\Gamma]$ are related to the Kazhdan-Lusztig polynomials via $p_{x,y} = (-1)^{l(x)+l(y)} v^{L(x)-L(y)} \overline{P_{x,y}^J}$.

4.9 Example:

Starting with an arbitrary $J \subseteq S$ and either the trivial or the sign module M for Ω_J , one obtains the module called M^J by Deodhar in [2]. The elements $p_{x,y} \in \mathbb{Z}[\Gamma] \Omega_J$ act on M by multiplication with polynomials similarly related to Deodhar's parabolic Kazhdan-Lusztig polynomials $P_{x,y}^J$ as the polynomials in the previous example are related to the absolute Kazhdan-Lusztig polynomials.

The fact that $m \mapsto 1|m$ is an injective map $M \rightarrow \text{HY}_J^S(M)$ provides a simple proof to conjecture [8, 4.2.23] from the author's thesis.

4.10 Corollary:

Let k be a commutative ring. Then the parabolic morphism $j : k\Omega_J \rightarrow k\Omega, e_s \mapsto e_s, x_{s,\gamma} \mapsto x_{s,\gamma}$ is injective.

Proof. Consider the Howlett-Yin-induction $M := \text{HY}_J^S(\Omega_J)$ of the regular $k\Omega_J$ -module. It is a $k\Omega$ -module so that $f : k\Omega \rightarrow M, a \mapsto a \cdot 1|1$ is a morphism of $k\Omega$ -left-modules. For all $s \in J$ one finds $f(j(e_s)) = 1|e_s$ and $f(j(x_{s,\gamma})) = 1|x_{s,\gamma}$ so that $f(j(a)) = 1|a$ holds for all $a \in k\Omega_J$. In particular we find that $f \circ j$ is injective so that j is injective. \square

We will therefore suppress the embedding altogether and consider $k\Omega_J$ as a true sub-algebra of $k\Omega$ from now on.

The Howlett-Yin-induction also provides the correction to a small error in corollary [8, 4.2.19] of the author's thesis.

4.11 Proposition:

Let k be a commutative ring. Define $E_J := \prod_{s \in J} e_s \prod_{s \in S \setminus J} (1 - e_s) \in k\Omega_J$. This element is non-zero in $k\Omega_J$.

The false proof in my thesis considered the Kazhdan-Lusztig- W -Graph KL^S and assuming falsely that each $J \subseteq S$ occurs as a left descent set $D_L(w)$ for some $w \in W$ I concluded that E_J must act non-trivially on this W -graph. This only works for finite Coxeter groups because a subset $J \subseteq S$ in fact occurs as a left descent set iff W_J is finite (c.f. [2, lemma 3.6]). In particular S itself does not occur as a left descent set if W is infinite. Nevertheless S occurs in the W -graph of the sign representation and $E_S \in k\Omega$ is therefore non-zero. This is the idea of the following proof:

Proof. Consider the sign representation $M = k \cdot m_0$ of $k\Omega_J$, i.e. $e_s m_0 = m_0$, $x_s m_0 = 0$ for all $s \in J$. The element $1|m_0 \in \text{HY}_J^S(M)$ satisfies:

$$e_s 1|m_0 = \begin{cases} 1|m_0 & s \in J \\ 0 & s \notin J \end{cases}$$

for all $s \in S$ so that $E_J \cdot 1|m_0 = 1|m_0$ and therefore $E_J \neq 0$. \square

5 Categorical properties of Howlett-Yin-induction

5.1 Theorem:

Let M, M_1, M_2 be $k\Omega_J$ -modules, and $\phi : M_1 \rightarrow M_2$ a $k\Omega_J$ -linear map.

a.) Using the notation from the preceding theorem, the map

$$\text{HY}_J^S(\phi) : \text{HY}_J^S(M_1) \rightarrow \text{HY}_J^S(M_2), z|m_1 \mapsto z|\phi(m_1)$$

is Ω -linear. In particular HY_J^S is a functor $k\Omega_J\text{-Mod} \rightarrow k\Omega\text{-Mod}$.

b.) $\text{HY}_J^S(\phi)$ commutes with the two canonicalisations, that is diagram 2 commutes.

$$\begin{array}{ccc} \text{Res}_H^\Omega \text{HY}_J^S(M_1) & \xrightarrow{\text{id} \otimes \text{HY}_J^S(\phi)} & \text{Res}_H^\Omega \text{HY}_J^S(M_2) \\ \downarrow c_{M_1} & & \downarrow c_{M_2} \\ \text{Ind}_{H_J}^H \text{Res}_{H_J}^{\Omega_J} M_1 & \xrightarrow{\text{Ind}_{H_J}^H \phi} & \text{Ind}_{H_J}^H \text{Res}_{H_J}^{\Omega_J} M_2 \end{array}$$

Figure 2: Functoriality of Howlett-Yin induction

In other words: The canonicalisation c is a natural isomorphism $\text{Res}_H^\Omega \circ \text{HY}_J^S \rightarrow \text{Ind}_{H_J}^H \circ \text{Res}_{H_J}^{\Omega_J}$.

Proof. That $\text{HY}_J^S(\phi)$ is Ω -linear is readily verified with the definition of the Ω -action.

$\text{Ind}_{H_J}^H(\phi)$ is certainly H -linear. Therefore $\phi(T_x \otimes M_1) = T_x \otimes \phi(M_1) \subseteq T_x \otimes M_2$ holds for all $x \in D_J$. It also commutes with ι . By functoriality of canonicalisation, the diagram 2 commutes. \square

5.2 Lemma:

The functor HY_J^S is exact, commutes with direct sums and satisfies

- a.) There is a sub- Ω - Ω_J -bimodule $\mathfrak{I}_J^S \leq \Omega$ such that $\bar{a} \otimes m \mapsto a \cdot 1|m$ is a natural isomorphism $\Omega/\mathfrak{I}_J^S \otimes_{\Omega_J} M \rightarrow \text{HY}_J^S(M)$.
- b.) $\text{HY}_J^S(M)$ has the following universal mapping property in $k\Omega$ -**mod**:

$$\text{Hom}(\text{HY}_J^S(M), X) \cong \{ f : M \rightarrow \text{Res}_{\Omega_J}^\Omega(X) \mid \mathfrak{I}_J^S \cdot f(M) = 0 \}$$

where the isomorphism is given by $F \mapsto (m \mapsto F(1|m))$.

Proof. It is clear from the definition that HY_J^S is exact and commutes with direct sums.

The Eilenberg-Watts-theorem (which characterizes cocontinuous and (right)exact functors between module categories, c.f. [3], [14]) implies $\text{HY}_J^S \cong Q \otimes_{\Omega_J} -$ for some Ω - Ω_J -bimodule Q . In fact the proof is constructive. It shows that one can choose Q to be $\text{HY}_J^S(\Omega_J)$ and the isomorphism as $\text{HY}_J^S(\Omega_J) \otimes M \rightarrow \text{HY}_J^S(M), z|a \otimes m \mapsto z|am$. Furthermore $\text{HY}_J^S(\Omega_J)$ is generated by the element $1|1$: The Ω_J -submodule generated by $1|1$ is $1|\Omega_J$ and in general $1|M$ generates $\text{HY}_J^S(M)$ as an Ω -module. Therefore $\text{HY}_J^S(\Omega_J)$ is isomorphic to some quotient Ω/\mathfrak{I}_J^S via $\bar{a} \mapsto a \cdot 1|1$.

The universal property follows from this presentation of the functor: Tensor the exact sequence $\mathfrak{I}_J^S \rightarrow \Omega \rightarrow \Omega/\mathfrak{I}_J^S \rightarrow 0$ with M . Right exactness of $- \otimes M$ implies that

$$\mathfrak{I}_J^S \otimes_{\Omega_J} M \rightarrow \Omega \otimes_{\Omega_J} M \rightarrow \text{HY}_J^S(M) \rightarrow 0$$

is exact. This provides a universal property of $\text{HY}_J^S(M)$ as the quotient of $\Omega \otimes M$ modulo the image of $\mathfrak{I}_J^S \otimes M \rightarrow \Omega \otimes M$. Combining this with Hom-tensor-adjunction $\text{Hom}(\Omega \otimes M, X) \cong \text{Hom}(M, \text{Res}_{\Omega_J}^\Omega(X))$ we obtain the result. \square

5.3 Proposition:

Let V_1, V_2 be two $k\Omega$ -modules and $f : V_1 \rightarrow V_2$ a k -linear map.

The f is $k\Omega$ -linear iff the induced map $\mathcal{A} \otimes_k V_1 \rightarrow \mathcal{A} \otimes_k V_2$ is kH -linear and $f(e_s m) = e_s f(m)$ holds for all $m \in V_1$.

Proof. Because $T_s = -v_s^{-1}e_s + v_s e_s + x_s$ the assumptions imply $f(x_s m) = x_s f(m)$ as elements of $\mathcal{A} \otimes V_2 = \bigoplus_\gamma v^\gamma V_2$. Now by definition $x_s = \sum_\gamma x_{s,\gamma} v^\gamma$ so that $\sum_\gamma f(x_{s,\gamma} m) v^\gamma = \sum_\gamma x_{s,\gamma} f(m) v^\gamma$. Comparing coefficients gives Ω -linearity. The reverse implication is clear. \square

Having a concept of induction makes it obvious to ask if this is transitive which turns out to be true as the next theorem shows.

5.4 Theorem:

Howlett-Yin-Induction is transitive. More precisely: If $J \subseteq K \subseteq S$, then

$$\tau_M : \text{HY}_K^S(\text{HY}_J^K(M)) \rightarrow \text{HY}_J^S(M), w|z|m \mapsto wz|m$$

is a natural $k\Omega$ -module isomorphism.

Proof. Consider the diagram 3. Here $t : \text{Ind}_{H_K}^{H_S} \circ \text{Ind}_{H_J}^{H_K} \rightarrow \text{Ind}_{H_J}^{H_S}$ is the natural isomorphism mapping $h_1 \otimes (h_2 \otimes m) \mapsto h_1 h_2 \otimes m$.

$$\begin{array}{ccc}
\text{Res}_{H_S}^{\Omega_S} \text{HY}_K^S \text{HY}_J^K M & \xrightarrow{\text{id}_A \otimes \tau_M} & \text{Res}_{H_S}^{\Omega_S} \text{HY}_J^S M \\
\downarrow c_{\text{HY}_J^K(M)} & & \downarrow c_M \\
\text{Ind}_{H_K}^{H_S} \text{Res}_{H_K}^{\Omega_K} \text{HY}_J^K M & & \\
\downarrow \text{Ind}_{H_K}^{H_S}(c_M) & & \\
\text{Ind}_{H_K}^{H_S} \text{Ind}_{H_J}^{H_K} \text{Res}_{H_J}^{\Omega_J} M & \xrightarrow{t_M} & \text{Ind}_{H_J}^{H_S} \text{Res}_{H_J}^{\Omega_J} M
\end{array}$$

Figure 3: Transitivity of Howlett-Yin-Induction

We will show that this diagram commutes. Note that t and c are natural H -linear isomorphisms. In particular this expresses $\text{id}_A \otimes \tau_M$ as a composition of H_S -linear natural isomorphism. We will easily verify that τ_M is in fact Ω_S -linear so that τ_M really is a natural isomorphism between Ω_S -modules.

To prove the diagram commutes we will show that the counter-clockwise composition of arrows from $\text{Res}_{H_S}^{\Omega_S} \text{HY}_J^S M$ to $\text{Ind}_{H_J}^{H_S} \text{Res}_{H_J}^{\Omega_J} M$ equals the canonicalisation c_M .

First note that all maps in the diagram are in fact $\hat{\mathcal{A}}$ -linear: $\text{id}_A \otimes \tau_M$ is trivially $\hat{\mathcal{A}}$ -linear, $c_{\text{HY}_J^K(M)}$ is because it is a canonicalisation, $\text{Ind}_{H_K}^{H_S}(c_M)$ is because c_M is and $\text{Ind}_{H_K}^{H_S}$ maps $\hat{\mathcal{A}}$ -linear maps to $\hat{\mathcal{A}}$ -linear maps and that $t_{\text{Res}_{H_K}^{\Omega_K}(M)}$ is also $\hat{\mathcal{A}}$ -linear can readily be verified.

Next we prove that the counter-clockwise composition maps $xy|m$ into $T_{xy} \otimes m + \sum_{w \in D_J^S} T_w \otimes \mathcal{A}_{>0} M$ for all $(x, y) \in D_K^S \times D_J^K$:

$$\begin{aligned}
1 \otimes xy|m & \xrightarrow{\text{id} \otimes \tau_M^{-1}} 1 \otimes x|y|m \\
& \xrightarrow{c_{\text{HY}_J^K(M)}} T_x \otimes y|m + \sum_{u \in D_K^S} T_u \otimes \mathcal{A}_{>0} \text{HY}_J^K(M)
\end{aligned}$$

$$\begin{aligned}
& \xrightarrow{\text{Ind}_{H_K}^{H_S}(c_M)} T_x \otimes T_y \otimes m + \sum_{(u,v) \in D_K^S \times D_J^K} T_u \otimes T_v \otimes \mathcal{A}_{>0} M \\
& \xrightarrow{t_{\text{Res}_{H_J}^{\Omega_J}(M)}} T_{xy} \otimes m + \sum_{uv \in D_J^S} T_{uv} \otimes \mathcal{A}_{>0} M
\end{aligned}$$

Theorem 2.7 shows that c_M is the only $\hat{\mathcal{A}}$ -linear map that maps $1 \otimes xy|m$ into $T_{xy} \otimes m + \sum_{w \in D_J^S} T_w \otimes \mathcal{A}_{>0} M$. This completes our proof that $\text{id}_{\mathcal{A}} \otimes \tau_M$ is H -linear and natural in M .

Furthermore $\tau_M(e_s x|y|m) = e_s \tau_M(x|y|m)$ follows directly from lemma 4.2 and the definition of the Ω -action so that τ_M is Ω -linear by proposition 5.3. \square

The next obvious question is if there is a Mackey decomposition for the Howlett-Yin-induction. There is no reason to expect that $\text{HY}_J^S(M)$ decomposes into a direct sum over double cosets since Ω unlike $\mathbb{Z}[W]$ and H does not have such a direct sum decomposition (and in fact $H_J^S(M)$ can be an indecomposable Ω -module). Instead we find a filtration indexed by the double cosets whose quotients play the role of the direct summands in the Mackey decomposition.

5.5 Lemma:

Let $J, K \subseteq S$. Define $D_{KJ} := D_K^{-1} \cap D_J$. Then

- a.) D_{KJ} is a system representatives of $W_K W_J$ -double cosets in W . More precisely $d \in D_{KJ}$ iff it is the unique element of minimal length within its double coset.
- b.) $D_J^S = \coprod_{d \in D_{KJ}} D_{K \cap dJ}^K \cdot d$.
- c.) For $d \in D_{KJ}$ and $x \in D_{K \cap dJ}^K$:

$$D_J^*(xd) \cap K = D_{K \cap dJ}^*(x) \cap K$$

where $*$ $\in \{+, 0, -\}$.

Proof. See [6, 2.1.6–2.1.9] \square

5.6: Fix some $d \in D_{KJ}$. For any Ω_J -module M one can define an $\Omega_{K \cap dJ}$ -module ${}^d M$ by

$$e_s \cdot {}^d m := {}^d(e_s {}^d m) \quad \text{and} \quad x_s \cdot {}^d m := {}^d(x_s {}^d m).$$

Similarly for any H_J -module V one can define a $H_{K \cap dJ}$ -module ${}^d V$ by

$$T_s \cdot {}^d v := {}^d(T_s {}^d v).$$

Note that ${}^d V$ is isomorphic to the $H_{K \cap dJ}$ -submodule $T_d \otimes V \subseteq \text{Ind}_{H_J}^{H_S} V$.

5.7 Theorem:

Let $J, K \subseteq S$. Furthermore let M be a $k\Omega_J$ -module and $V := \text{Res}_{H_J}^{\Omega_J} M$ its associated kH_J -module. For all $d \in D_{KJ}$ define the following k -submodules of $\text{HY}_J^S(M)$ and $\text{Ind}_{H_J}^{H_S}(V)$ respectively:

$$F^{\leq d} \text{HY}_J^S(M) := \sum_{\substack{a \in D_{KJ}, a \leq d \\ w \in D_{K \cap aJ}^K}} wa|M$$

$$F^{\leq d} \text{Ind}_{H_J}^{H_S}(V) := \sum_{\substack{a \in D_{KJ}, a \leq d \\ w \in D_{K \cap aJ}^K}} T_{wa} \otimes V$$

The following hold for all $d \in D_{KJ}$:

- a.) $F^{\leq d} \text{HY}_J^S(M)$ is a Ω_K -submodule, $F^{\leq d} \text{Ind}_{H_J}^{H_S}(V)$ is a H_K - and $\hat{\mathcal{A}}$ -submodule and the canonicalisation map c_M identifies these with each other.
- b.) The map $\Psi_M^d : \text{HY}_{K \cap dJ}^K({}^d M) \rightarrow F^{\leq d} \text{HY}_J^S(M) / F^{< d} \text{HY}_J^S(M)$ which is defined by $w|{}^d m \mapsto wd|m$ for all $w \in D_{K \cap dJ}^K$ is a natural isomorphism of Ω_K -modules.

Proof. That $F^{\leq d} \text{HY}_J^S(M)$ is a Ω_K -submodule follows directly from the definition of the Ω -action on $\text{HY}_J^S(M)$ and the observation $w'd' \leq wd \implies d' \leq d$ for all $w, w' \in W_K$, $d, d' \in D_{KJ}$. That $F^{\leq d} \text{Ind}_{H_J}^{H_S}(V)$ is a H_K -submodule follows from the fact that it is equal to $\sum_{a \leq d} \text{span} \{ T_w \mid w \in W_K a W_J \} \otimes V$.

To prove the second claim, observe that the given map is certainly a k -linear bijection. Next we will show that it makes the diagram 4 commute.

$$\begin{array}{ccc} F^{\leq d} \text{HY}_J^S(M) / F^{< d} \text{HY}_J^S(M) & \xleftarrow{\Psi_M^d} & \text{HY}_{K \cap dJ}^K({}^d M) \\ \downarrow \overline{c_M} & & \downarrow c_{dM} \\ F^{\leq d} \text{Ind}_{H_J}^{H_S}(V) / F^{< d} \text{Ind}_{H_J}^{H_S}(V) & \xleftarrow{T_{xd} \otimes m \leftarrow T_x \otimes {}^d m} & \text{Ind}_{K \cap dJ}^{H_K}({}^d V) \end{array}$$

Figure 4: Mackey isomorphism for Howlett-Yin-Induction

This will again be done by utilising the uniqueness of the canonicalisation map. Observe that all maps involved in the diagram are $\hat{\mathcal{A}}$ -linear bijections. It is also readily verified that both the counter-clockwise and the clockwise compositions map $w|{}^d m$ into $T_{wd} \otimes m + \sum_{x \in D_{K \cap dJ}^K} T_{xd} \otimes \mathcal{A}_{>0} M$ so that the diagram indeed commutes by uniqueness of the canonicalisation map.

Because canonicalisation is H_K -linear, we conclude that Ψ_M^d is also H_K -linear. It follows directly from the above lemma that $\Psi_M^d(e_s \cdot w|{}^d m) = e_s \cdot \Psi_M^d(w|{}^d m)$ for $s \in K$. Proposition 5.3 implies again that Ψ_M^d is indeed Ω_K -linear. \square

6 Applications

Denote with $\mu_{x,z}^{s,J}$ the elements in $\mathbb{Z}[\Gamma]\Omega_J$ from definition 4.4 to make the dependence from $J \subseteq S$ explicit. Note that these elements do not depend on S in the sense that for any parabolic subgroup $W_J \subseteq W_K \subseteq W$ with $x, z \in W_K$ the elements computed w.r.t. the inclusion $J \subseteq S$ are the same as when computed w.r.t. the inclusion $J \subseteq K$.

6.1 Corollary:

$J \subseteq K \subseteq S$. Let $u, x \in D_K^S$, $v, y \in D_J^K$.

a.) If $u = x$, then $\mu_{xv,xy}^{s,J} = \begin{cases} \mu_{v,y}^{s^x,J} & s \in D_K^0(x) \\ 0 & \text{otherwise} \end{cases}$.

b.) If $u < x$, then $\sum_{v \in D_J^K} v|\mu_{uv,xy}^{s,J} = \mu_{u,x}^{s,K} \cdot y|1$ as elements of $\text{HY}_J^K(\Omega_J)$.

Proof. We consider the identification $\text{HY}_K^S \text{HY}_J^K(M) \cong \text{HY}_J^S(M)$ from theorem 5.4 and the action of x_s on both modules:

$$\begin{aligned} x_s \cdot xy|m &= \begin{cases} sxy|m + \sum_{uv} uv|\mu_{uv,xy}^{s,J}m & s \in D_J^+(xy) \\ xy|x_{s^x}y|m + \sum_{uv} uv|\mu_{uv,xy}^{s,J}m & s \in D_J^0(xy) \\ 0 & s \in D_J^-(xy) \end{cases} \\ x_s \cdot x|y|m &= \begin{cases} sx|y|m + \sum_{u < x} u|\mu_{u,x}^{s,K} \cdot y|m & s \in D_K^+(x) \\ x|x_{s^x} \cdot y|m + \sum_{u < x} u|\mu_{u,x}^{s,K} \cdot y|m & s \in D_K^0(x) \\ 0 & s \in D_K^-(x) \end{cases} \\ &= \begin{cases} sx|y|m + \sum_{u < x} u|\mu_{u,x}^{s,K} \cdot y|m & s \in D_K^+(x) \\ x|s^x y|m + \sum_{v < y} x|v|\mu_{v,y}^{s^x,J}m + \sum_{u < x} u|\mu_{u,x}^{s,K} \cdot y|m & s \in D_K^0(x), s^x \in D_J^+(y) \\ x|y|x_{s^x}y|m + \sum_{v < y} x|v|\mu_{v,y}^{s^x,J}m + \sum_{u < x} u|\mu_{u,x}^{s,K} \cdot y|m & s \in D_K^0(x), s^x \in D_J^0(y) \\ 0 & s \in D_K^0(x), s^x \in D_J^-(y) \\ 0 & s \in D_K^-(x) \end{cases} \end{aligned}$$

Therefore

$$\sum_{uv} u|v|\mu_{uv,xy}^{s,J}m = \begin{cases} \sum_{u < x} u|\mu_{u,x}^{s,K} \cdot y|m & s \in D_K^+(x) \\ \sum_{v < y} x|v|\mu_{v,y}^{s^x,J}m + \sum_{u < x} u|\mu_{u,x}^{s,K} \cdot y|m & s \in D_K^0(x), s^x \in D_J^+(y) \cup D_J^0(y) \\ 0 & \text{otherwise} \end{cases}$$

Comparing the component $u|\text{HY}_J^K(M)$, we find

$$\sum_v v|\mu_{uv,xy}^{s,J}m = \begin{cases} \mu_{u,x}^{s,K} \cdot y|m & u < x \\ \sum_{v < y} v|\mu_{v,y}^{s^x,J}m & u = x, s \in D_K^0(x) \\ 0 & \text{otherwise} \end{cases}$$

Now set $M := \Omega_J$ and $m := 1$. □

6.2: This suggests the following algorithm for computing the $\mu_{w,z}^{s,J}$ for all $w, z \in D_J^S$ and all $s \in S$:

1. Choose a flag $J = K_0 \subsetneq K_1 \subsetneq \dots \subsetneq K_n = S$.
2. For all $i = 1, \dots, n$, all $u, x \in D_{K_{i-1}}^{K_i}$, and all $s \in K_i$ compute $\mu_{u,x}^{s,K_{i-1}} \in \Omega_{K_{i-1}}$ with algorithm 4.7.
3. For $i = 1, \dots, n$ compute $\mu_{w,z}^{s,J}$ for all $w, z \in D_J^{K_i}$ and all $s \in K_i$:
 - 3.1. Write $w = uv$, $z = xy$ with $u, x \in D_{K_{i-1}}^{K_i}$ and $v, y \in D_J^{K_{i-1}}$.
 - 3.2. If $u \not\leq x$, then $\mu_{w,z}^{s,J} = 0$.
 - 3.3. If $u = x$, then $\mu_{w,z}^{s,J} = \begin{cases} \mu_{v,y}^{s^x,J} & s \in D_{K_{i-1}}^0(x) \\ 0 & \text{otherwise} \end{cases}$.
 - 3.4. If $u < x$, then write the action of $\mu_{u,x}^{s,K_{i-1}} \in \Omega_{K_{i-1}}$ on $\text{HY}_J^{K_{i-1}}(\Omega_J)$ as a $D_J^{K_{i-1}} \times D_J^{K_{i-1}}$ -matrix with entries in Ω_J . Then $\mu_{w,z}^{s,J}$ equals the (v, y) -th entry of this matrix.

Note that the action of $\Omega_{K_{i-1}}$ on $\text{HY}_J^{K_{i-1}}(\Omega_J)$ only involves values of $\mu_{w',z'}^{s',J}$ where $w', z' \in D_J^{K_{i-1}}$ and $s' \in K_{i-1}$ which are already known by the previous iteration of the loop.

The big advantage of this algorithm over a direct computation of all $\mu_{w,z}^{s,J}$ with algorithm 4.7 is that it the expensive recursion over D_J^S is replaced by n collectively cheaper recursions over $D_{K_0}^{K_1}, D_{K_1}^{K_2}, \dots, D_{K_{n-1}}^{K_n}$ so that fewer polynomials $p_{w,z}$ need to be computed and the computed elements are less complex (measured for example by the maximal length of occurring words in the generators e_s, x_s of Ω) and therefore need less memory.

Additionally the n calls to algorithm 4.7 in step 2 are independent of each other and can be executed in parallel which can lead to a sizeable speed-up.

6.3: The Mackey-isomorphism Ψ^d from theorem 5.7 translates into the equation

$$\mu_{yd,wd}^{s,J} = \kappa_d(\mu_{y,w}^{s,K \cap^d J})$$

for all $d \in D_{KJ}$, all $y, w \in D_{K \cap^d J}^K$, and all $s \in K$ where $\kappa_d : \Omega_{K \cap^d J} \rightarrow \Omega_{K^d \cap J}$ is the isomorphism $e_s \mapsto e_{s^d}, x_s \mapsto x_{s^d}$.

Provided one knows all $\mu_{u,v}^{s,T}$ for all $T \subseteq K$, all $u, v \in D_T^K$, and all $s \in K$, one can therefore use that knowledge to partially calculate $\mu_{y,w}^{s,J}$. This in turn might be used to give the recursion from algorithm 4.7 a head start and reduce the necessary recursion depth.

6.4: Recall the definition of Kazhdan-Lusztig-(left)cells: Define a preorder $\preceq_{\mathcal{L}}$ on W by defining $\{y \in W \mid y \preceq_{\mathcal{L}} z\}$ to be the smallest subset $\mathfrak{C} \subseteq W$ such that the subspace

$\text{span}_{\mathbb{Z}[\Gamma]} \{ C_y \mid y \in \mathfrak{C} \}$ is a H -submodule of H . The preorder then defines an equivalence relation $\sim_{\mathcal{L}}$ as usual by $x \sim_{\mathcal{L}} y : \iff x \preceq_{\mathcal{L}} y \wedge y \preceq_{\mathcal{L}} x$. The equivalence classes of this relation are called *left cells*.

Transitivity of Howlett-Yin-induction also enables us to effortlessly reprove a result of Geck regarding the induction of cells (see [4]):

6.5 Corollary: a.) $\mathfrak{C} \subseteq W$ is $\preceq_{\mathcal{L}}$ -downward closed iff $\text{span}_{\mathbb{Z}} \{ x|1 \mid x \in \mathfrak{C} \}$ is a Ω -submodule of $\text{HY}_{\emptyset}^S(\Omega_{\emptyset})$.

b.) If $\mathfrak{C} \subseteq W_J$ is a union of left cells, then $D_J^S \cdot \mathfrak{C} \subseteq W$ is also a union of left cells.

Proof. Consider the Kazhdan-Lusztig- W -graph $\text{HY}_{\emptyset}^S(\Omega_{\emptyset})$. Then $\{ x|1 \mid x \in W \}$ constitute a \mathbb{Z} -basis of this module which (under the canonicalisation map) corresponds to the basis $\{ C_x \mid x \in W \}$. That \mathfrak{C} is a $\preceq_{\mathcal{L}}$ -downward closed means that $\text{span}_{\mathbb{Z}[\Gamma]} \{ C_x \mid x \in \mathfrak{C} \}$ is a H -submodule of H . Because $e_s \cdot x|1 \in \{ 0, x|1 \}$ for all x and s , every subset of the form $\text{span}_{\mathbb{Z}} \{ x|1 \mid x \in \mathfrak{C} \}$ is closed under multiplication with e_s . Since e_s and T_s together generated Ω this proves the first statement.

Now let $\mathfrak{C} \subseteq W_J$ be $\preceq_{\mathcal{L}}$ -downward closed and $M := \text{span}_{\mathbb{Z}} \{ x|1 \mid x \in \mathfrak{C} \}$ be the corresponding submodule of $\text{HY}_{\emptyset}^J(\Omega_{\emptyset})$. Then $\text{HY}_J^S(M) = \text{span}_{\mathbb{Z}} \{ w|x|1 \mid w \in D_J^S, x \in \mathfrak{C} \}$ is a submodule of $\text{HY}_J^S \text{HY}_{\emptyset}^J(\Omega_{\emptyset}) \cong \text{HY}_{\emptyset}^S(\Omega_{\emptyset})$. In other words $D_J^S \cdot \mathfrak{C}$ is a $\preceq_{\mathcal{L}}$ -downward closed set of W . Because every union of cells can be written as a set difference $\mathfrak{C}_1 \setminus \mathfrak{C}_2$ for some downward closed sets $\mathfrak{C}_2 \subseteq \mathfrak{C}_1 \subseteq W$, this proves the second statement. \square

6.6: Modifying the above algorithm 6.2 such that only μ -values for elements in $D_J^S \cdot \mathfrak{C}$ are computed, we recover Geck's PyCox algorithm for the decomposition into left cells.

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